A REFINEMENT OF THE KARMAN-POHLHAUSEN INTEGRAL METHOD IN BOUNDARY LAYER THEORY

V. N. Volkov

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A basis relation for boundary layer calculations has been obtained by double integration of the Prandtl equations. It is shown that this method leads to more accurate results than the Karman-Pohlhausen method.

The Karman-Pohlhausen method is based on the integral momentum relation (1):

$$\frac{d}{dx}\int_0^\delta u(U-u)\,dy+\frac{dU}{dx}\int_0^\delta (U-u)\,dy=\nu\left(\frac{\partial u}{\partial y}\right)_0. \quad (1)$$

The method involves replacement of the unknown exact velocity distribution in the layer by a specially chosen distribution satisfying the boundary conditions of the problem. It is simple and in a number of cases, moreover, gives quite accurate results. Its effectiveness is evidently due to the fact that the values of the integrals on the left side in (1) do not change much if the subintegral function departs a little from the true value, provided the boundary conditions are observed. The main source of error of the method is associated with the right side of (1), where there is a derivative of the unknown function. When this function is replaced by another, arbitrary one, the error in the derivative may be very considerable. It is for this reason that, in a region of pressure increase, and in particular, in locating the separation point of the layer, the Karman-Pohlhausen method gives quite unreliable results (leading in some cases to clearly erroneous conclusions, even in a qualitative sense [2]).

The method will be appreciably improved if the right side of (1) is also put in the form of integrals of the unknown function u. This may be effected by double integration of the original Prandtl equations.

The differential equations of the boundary layer have the form

$$\nu\frac{\partial^2 u}{\partial y^2}=u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-U\frac{dU}{dx},$$
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \quad (2)$$

Single integration of the equations of (2) yields

$$\nu\frac{\partial u}{\partial y}-\nu\left(\frac{\partial u}{\partial y}\right)_0=\frac{\partial}{\partial x}\int_0^y u^2dy-u\frac{\partial}{\partial x}\int_0^y u\,dy-U\frac{dU}{dx}y. \quad (3)$$

If we put $y=\infty$ in (3), it is not difficult to obtain relation (1). Integrating (3) once more, we have

$$\nu u-\nu\left(\frac{\partial u}{\partial y}\right)_0 y=\int_0^y dy\frac{\partial}{\partial x}\left(\int_0^y u^2dy\right)-$$
$$-\int_0^y u\,dy\frac{\partial}{\partial x}\left(\int_0^y u\,dy\right)-U\frac{dU}{dx}\frac{y^2}{2}. \quad (4)$$

We take into consideration, as was done also in (1), the boundary condition of the layer $\delta(x)$.

Putting $y=\delta(x)$, and substituting the expression found from (4) for $\nu\left(\frac{\partial u}{\partial y}\right)_0$ into (3), we have in place of (1) the equation

$$\frac{d}{dx}\int_0^\delta u^2dy-U\frac{d}{dx}\int_0^\delta u\,dy-\frac{1}{2}U\frac{dU}{dx}\delta=$$
$$=-\frac{\nu U}{\delta}-\frac{1}{\delta}\int_0^\delta u\,dy\frac{\partial}{\partial x}\left(\int_0^y u\,dy\right)+ \quad (5)$$
$$+\frac{1}{\delta}\int_0^\delta dy\frac{\partial}{\partial x}\left(\int_0^y u^2dy\right).$$

FLAT IMPERMEABLE PLATE

Since the velocity profile in this case has similarity $\left(u=U_0 f\left(\frac{y}{\delta(x)}\right)\right)$, it is easy to put (5) in the form

$$A\frac{d\delta^2}{dx}=\frac{\nu}{U_0}, \quad (6)$$

where

$$A=\frac{1}{2}\left(\int_0^1 d\eta\int_0^\eta f^2d\eta-\int_0^1 fd\eta\,\times\right.$$
$$\left.\times\int_0^\eta fd\eta-\int_0^1 f^2d\eta+\int_0^1 fd\eta\right).$$

Therefore,

$$\delta=A^{-1/2}\sqrt{\nu x/U_0}. \quad (7)$$

We find the friction stress on the plate from the formula

$$\frac{\tau_0}{\rho} = \nu\left(\frac{\partial u}{\partial y}\right)_0 = -\frac{d}{dx}\int_0^\delta u^2 dy + U_0\frac{d}{dx}\int_0^\delta u\,dy = BU_0\frac{d\delta}{dx}, \tag{8}$$

where

$$B = \int_0^1 f\,d\eta - \int_0^1 f^2\,d\eta.$$

According to (7) we obtain

$$\frac{\tau_0}{\rho} = \frac{B}{2\sqrt{A}}\sqrt{\frac{\nu U_0^3}{x}}. \tag{9}$$

We will compare the results of calculating a boundary layer by the Karman-Pohlhausen method with those of the refined method. The table shows values of δ and τ_0 calculated for various distribution functions according to both methods. Comparison indicates that the values of δ and τ_0 calculated by the refined method are, firstly, closer to the true values than are those from the Karman-Pohlhausen method, and, secondly and this is more important in the given case the refined values of δ and τ_0 give less scatter and are less sensitive to the choice of type of distribution function $f(\eta)$.

GENERAL CASE OF AN IMPERMEABLE WALL

In the general case the velocity profile losses the property of similarity. Following the Karman-Pohlhausen method, we take into account the shape-factor

$$\lambda = \frac{\delta^2}{\nu}\frac{dU}{dx} \tag{10}$$

and put the solution of (5) in the form

$$u = U\left[f\left(\frac{y}{\delta}\right) + \lambda\varphi\left(\frac{y}{\delta}\right)\right]. \tag{11}$$

Substitution of function (11) into (5) leads, after a number of simple transformations, to the equation

$$M\frac{d\delta^2}{dx} + N\delta^2 = \nu, \tag{12}$$

where M and N are functions of U, dU/dx, λ and $d\lambda/dx$.

If the shape-factor is taken as the basic variable being sought, Eq. (12) becomes

$$\dot{\lambda}[A_1 + (A_{12} + C) + (A_2 + B_2)\lambda^2] - \lambda\frac{\ddot{U}}{\dot{U}}(A_1 + A_{12}\lambda + A_2\lambda^2) + \lambda\frac{\dot{U}}{U}(B_1 + B_{12}\lambda + B_2\lambda^2) = \frac{\dot{U}}{U}\,^{*)}, \tag{13}$$

where A_i, B_i, A_{12}, B_{12}, C are certain universal numbers,

$$A_1 = (1/2)(\varepsilon_1 - \gamma_1 + \alpha_1 - \beta_1),\quad A_2 = (1/2)(\alpha_2 - \gamma_2 - \beta_2),$$

$$A_{12} = (1/2)(\varepsilon_2 - 2\gamma_{12} + 2\alpha_{12} - \beta_{12} - \beta_{21}),$$

$$B_1 = 1/2 + \varepsilon_1 - 2\gamma_1 + 2\alpha_1 - \beta_1, \tag{14}$$

$$B_{12} = \varepsilon_2 - 4\gamma_{12} + 4\alpha_{12} - \beta_{12} - \beta_{21},$$

$$B_2 = 2\alpha_2 - 2\gamma_2 - \beta_2,\quad C = \varepsilon_2 - 2\gamma_{12} + 2\alpha_{12} - \beta_{12}.$$

The numbers α, β, γ, ε in (14) are found by simple integration of functions f and φ:

$$\varepsilon_1 = \int_0^1 f\,d\eta,\quad \gamma_1 = \int_0^1 f^2 d\eta,\quad \beta_1 = \int_0^1 f\,d\eta\int_0^\eta f\,d\eta,$$

$$\alpha_1 = \int_0^1 d\eta\int_0^\eta f^2 d\eta,\quad \varepsilon_2 = \int_0^1 \varphi\,d\eta,\quad \gamma_2 = \int_0^1 \varphi^2 d\eta, \tag{15}$$

$$\beta_2 = \int_0^1 \varphi\,d\eta\int_0^\eta \varphi\,d\eta,\quad \alpha_2 = \int_0^1 d\eta\int_0^\eta \varphi^2 d\eta,\quad \gamma_{12} = \int_0^1 f\varphi\,d\eta,$$

$$\beta_{21} = \int_0^1 \varphi\,d\eta\int_0^\eta f\,d\eta,\quad \beta_{12} = \int_0^1 f\,d\eta\int_0^\eta \varphi\,d\eta,\quad \alpha_{12} = \int_0^1 d\eta\int_0^\eta \varphi f\,d\eta.$$

The function φ is a correction to f, and so the integrals from φ are considerably less than those from f, i.e., the numbers ε_1, γ_1, β_1, α_1, are considerably greater than the remaining numbers. For the same reason A_1 and B_1 are considerably greater than the numbers A_2, A_{12}, B_2, B_{12}, C (this may be verified directly in the examples). Taking this into consideration, Eq. (13) may be integrated approximately, after first omitting terms containing A_2, A_{12}, B_2, B_{12}, C. This leads to the simple linear equation

$$A_1\left(\dot{\lambda}_1 - \lambda_1\frac{\ddot{U}}{\dot{U}}\right) + B_1\lambda_1\frac{\dot{U}}{U} = \frac{\dot{U}}{U}. \tag{16}$$

*The dot denotes differentiation with respect to x.

Values of δ and τ_0 Calculated for Various Distribution Functions

Distribution $f(\eta)$	$\delta/\sqrt{\nu x/U_0}$		$\frac{\tau_0}{\rho}/\sqrt{\nu U_0^3/x}$	
	by Karman-Pohlhausen method	by improved method	by Karman-Pohlhausen method	by improved method
$f=\eta$	3.464	4	0.289	0.333
$f=\frac{3}{2}\eta-\frac{1}{2}\eta^3$	4.64	4.825	0.323	0.336
$f=2\eta-2\eta^3+\eta^4$	5.84	5.7	0.343	0.336
$f=\sin\frac{\pi}{2}\eta$	4.8	4.911	0.327	0.335
Exact	5		0.332	

This kind of approximation (we shall call it the first approximation) corresponds to the case $\varphi = 0$.

The solution of (16) has the form

$$\lambda_1 = \frac{\dot{U}}{A_1 U^{B_1/A_1}} \int_0^x U^{B_1/A_1 - 1}\, dx. \tag{17}$$

The friction stress in the first approximation is calculated from the formula

$$\frac{\tau_0}{\rho} = \frac{\nu}{\delta}\frac{U^2}{\dot{U}}\left\{A_1'\left(\dot{\lambda}_1 - \frac{\ddot{U}}{\dot{U}}\lambda_1\right) + B_1'\frac{\dot{U}}{U}\lambda_1\right\}, \tag{18}$$

where

$$A_1' = (\varepsilon_1 - \gamma_1)/2,\quad B_1' = 1 + \varepsilon_1 - 2\gamma_1.$$

Using (18), we find the following equation for the separation point of the layer:

$$\lambda_1 = \lambda_0 = -\frac{A_1'}{A_1}\Big/\left(B_1' - B_1\frac{A_1'}{A_1}\right). \tag{19}$$

Therefore, in the first approximation the value λ_1 at the separation point is some universal number, independent of U(x). This fact may be used in the choice of the function $\varphi(x)$. We have to choose φ such that

$$\text{when}\quad \lambda = \lambda_0\quad \left(\frac{\partial u}{\partial y}\right)_0 = 0. \tag{20}$$

In addition, the usual boundary conditions are imposed on φ, namely:

$$\left(\frac{d^2\varphi}{d\eta^2}\right)_0 = -1,\quad \varphi(1) = 0,\quad \varphi(0) = 0,\quad \left(\frac{\partial\varphi}{\partial\eta}\right)_1 = 0. \tag{21}$$

For example, if $f = \eta$, according to (19), $\lambda_0 = -4$, and φ should be chosen, in accordance with (20) and (21), in the form

$$\varphi = (1/4)(\eta - 2\eta^2 + \eta^3).$$

The calcualtions show that for the function $f = 2\eta - 2\eta^3 + \eta^4$, $\lambda_0 = -8.95$. Assuming also in this case that $\lambda_0 = -9$, φ must be chosen in the form

$$\varphi = (1/18)(4\eta - 9\eta^2 + 6\eta^3 - \eta^4).$$

In the second approximation

$$\lambda_2 = \lambda_1 + \lambda'. \tag{22}$$

Here λ' is a small correction to the first approximation.

We substitute λ_1 in the nonlinear terms of (13) and write the equation relative to λ':

$$A_1\dot{\lambda}' - [A_{12} + C + (A_2 + B_2)\lambda_1]\lambda_1\dot{\lambda}_1 + \left(B_1\frac{\dot{U}}{U} - A_1\frac{\ddot{U}}{\dot{U}}\right)\lambda' - \left[(A_{12} + A_2\lambda_1)\frac{\ddot{U}}{\dot{U}} - (B_{12} + B_2\lambda)\frac{\dot{U}}{U}\right]\lambda^2 = 0. \tag{23}$$

Calculations show that A_2 and B_2 are considerably less than A_{12}, B_{12}, C. Therefore terms with A_2 and B_2 may be omitted in the second approximation. Then, taking (16) into account, we find the solution of (23) in the form

$$\lambda' = \frac{\dot{U}}{A_1 U^{B_1/A_1}}\int_0^x \left\{\left[\left(\frac{B_1}{A_1}(A_{12} + C) - B_{12}\right)\cdot \frac{\dot{U}}{U} - A_1\frac{\ddot{U}}{\dot{U}}\right]\lambda_1^2 - \frac{A_{12} + C}{A_1}\frac{\dot{U}}{U}\lambda_1\right\}\frac{U^{B_1/A_1}}{\dot{U}}dx. \tag{24}$$

Thus, in the second approximation the problem is reduced to carrying out two successive quadratures (17) and (24).

Knowing $\lambda(x)$, it is not hard to find τ_0/ρ and the separation point of the layer. This separation point is determined from the algebraic equation

$$\lambda_0 = \lambda_1(x) + \lambda'(x). \quad (25)$$

It is evidently not hard to generalize the method suggested in the case of a boundary layer with outflow of fluid from the wall, and to extend it to the temperature and diffusion boundary layers.

As a very simple example, we will examine the flow of a fluid near the critical point. In this case $U = U_0x$.

Relations (17) and (18) easily allow us to find λ and τ_0/ρ in the first approximation:

$$\lambda_1 = B_1^{-1}, \quad \tau_0/\rho = B_1' \sqrt{B_1^{-1}}\, \nu^{1/2} U_0^{1/2} U. \quad (26)$$

In the case of $f(\eta) = \eta$ we have

$$\lambda_1 = 8/3, \quad \tau_0/\rho = 1.36\, \nu^{1/2} U_0^{1/2} U. \quad (27)$$

In the case $f(\eta) = 2\eta - 2\eta^3 + \eta^4$ we have

$$\lambda_1 = 5.96, \quad \tau_0/\rho = 1.31\nu^{1/2} U_0^{1/2} U. \quad (28)$$

The exact value is $\tau_0/\rho = 1.234\nu^{1/2} U_0{}^{1/2} U$, while according to the Karman-Pohlhausen method $\tau_0/\rho = 1.19\, \nu^{1/2} U_0^{1/2} U$ (1).

Thus, our method in this case leads to satisfactory results even in the first approximation.

In conclusion we note that the method described above borders upon Shvets' method (3). All the results obtained by that method (3-5) may also be obtained by our method. To show this we examine the first approximation in the very simple assumption that $f(\eta) = \eta$.

For example, the formula in (3)

$$\lambda = (16\dot{U}/U^6) \int_0^x U^5 dx$$

is a special case of the general formula (17), under the assumption that $f = \eta$.

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Lenin Power Engineering Institute, Ivanovo